Preprint typeset in JHEP style - HYPER VERSION

SU-4252-747
DIAS-02-12

Noncommutative Chiral Anomaly and The Dirac-Ginsparg-Wilson Operator

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Abstract: It is shown that the local axial anomaly in 2−dimensions emerges naturally if one postulates an underlying noncommutative fuzzy structure of spacetime . In particular the Dirac-Ginsparg-Wilson relation on $\mathbf{S}^2_F$ is shown to contain an edge effect which corresponds precisely to the "fuzzy" $U(1)_A$ axial anomaly on the fuzzy sphere . We also derive a novel gauge-covariant expansion of the quark propagator in the form $\frac{1}{\mathcal{D}_{AF}}=\frac{a\hat{\Gamma}^L}{2}+\frac{1}{\mathcal{D}_{Aa}}$ where $a=\frac{2}{2l+1}$ is the lattice spacing on $\mathbf{S}^2_F$, $\hat{\Gamma}^L$ is the covariant noncommutative chirality and $\mathcal{D}_{Aa}$ is an effective Dirac operator which has essentially the same IR spectrum as $\mathcal{D}_{AF}$ but differes from it on the UV modes. Most remarkably is the fact that both operators share the same limit and thus the above covariant expansion is not available in the continuum theory . The first bit in this expansion $\frac{a\hat{\Gamma}^L}{2}$ although it vanishes as it stands in the continuum limit , its contribution to the anomaly is exactly the canonical theta term. The contribution of the propagator $\frac{1}{\mathcal{D}_{Aa}}$ is on the other hand equal to the toplogical Chern-Simons action which in two dimensions vanishes identically .

Keywords: Local Anomaly , Fuzzy Physics , Noncommutative Field Theory and Geometry.

Contents

1. Introduction

Fuzzy physics [[2, 3] , see also [4] and references therein] , like lattice gauge theories, is aiming for a nonperturbative regularization of chiral gauge theories. Discretization in fuzzy physics is achieved by treating the underlying spacetimes as phase spaces then quantizing them in a canonical fashion which means in particular that we are effectively replacing the underlying spacetimes by non-commutative matrix models or fuzzy manifolds [4]. As a consequence , this regularization will preserve all symmetries and topological features of the problem . Indeed a fuzzy space is by construction a discrete lattice-like structure which serves to regularize, it allows for an exact chiral invariance to be formulated , but still the fermion-doubling problem is completely avoided [7] .

Global chiral anomalies on these models were , along with other topological non-trivial field configurations, formulated in [8, 9, 15, 19] , while local anomaly on fuzzy $\mathbf{S}_F^2$ was treated first in [6] then in [16, 17] . The relevance of Ginsparg-Wilson relations in noncommutative matrix models was noted first in [7] and [18] then in [16] . In this article we will show that despite the fact that the concept of point is lacking on fuzzy $\mathbf{S}^2$, we can go beyond global considerations and define a "fuzzy" axial anomaly associated with a "fuzzy" $U(1)$ global chiral symmetry.

The plan of the paper is as follows . Section 2 contains a brief description of fuzzy $\mathbf{S}^2$ and its star product. Fuzzy $U(1)$ gauge theory and fermion action on $\mathbf{S}^2_F$ are introduced in section 3 where we also show the absence of the fermion doubling problem on $\mathbf{S}^2_F$. The Dirac-Ginsparg-Wilson relation on $\mathbf{S}^2_F$ and the corresponding fuzzy chiral transformations as well as the "fuzzy" $U(1)_A$ axial anomaly are discussed in section 4. In Section 5 we derive a novel gauge-covariant expansion of the quark propagator and show explicitly that no analogous expansion exists in the continuum theory. The continuum limit is computed in this section where beside the canonical theta term we obtain the Chern-Simons action which vanishes identically in two dimensions. We conclude in section 6 .

2. The Fuzzy Sphere

2.1 Algebra

The $2-$dimensional continuum sphere is the co-adjoint orbit of $SU(2)$ through the Pauli matrix σ_3 and thus it admits a symplectic structure which can be quantized in a canonical fashion to give the so-called fuzzy sphere. We now explain briefly this result .

We can define $\mathbf{S}^2$ by the projector $P=\frac{\mathbf{1}}{2}+n_at_a$ with $t_a=\frac{\sigma_a}{2}$ since for example the requirement $P^2=P$ gives precisely the defining equation of $\mathbf{S}^2$ as a surface embedded in $\mathbf{R}^3$, namely $\vec{n}^2=1$. At the north pole of $\mathbf{S}^2$, namely at $\vec{n}_0=(0,0,1)$, this projector is $P_0=diag(1,0)$ which projects down to the state $|\vec{n}_0,\frac{1}{2}>=(1,0)$ of the $2-$dimensional Hilbert space $\mathbf{H}_{\frac{1}{2}}$ of the fundamental representation of $SU(2)$. A general point $\vec{n}$ of $\mathbf{S}^2$ is obtained by $\vec{n}=g\vec{n}_0$ where $g{\in}SU(2)$. The corresponding state in $\mathbf{H}_{\frac{1}{2}}$ is $|\vec{n},\frac{1}{2}>=g|\vec{n}_0,\frac{1}{2}>$. The projector on this state is $P=|\vec{n},\frac{1}{2}><\vec{n},\frac{1}{2}|=gP_0g^{+}$ provided

$$g\sigma_3g^{+}=n^a\sigma_a. \tag{2.1}$$

From this expression it is clear that $\mathbf{S}^2$ is indeed given by $SU(2)/U(1)$, and that points $\vec{n}{\in}\mathbf{S}^2$ are equivalence classes $\vec{n}=[g]=[gh]$ where $h{\in}U(1)$, i.e $\mathbf{S}^2$ is the co-adjoint orbit of $SU(2)$ through σ_3 .

$|\vec{n},\frac{1}{2}>$ and $|\vec{n}_0,\frac{1}{2}>$ are the "fundamental" coherent states at $\vec{n}$ and $\vec{n}_0$ respectively. Roughly speaking these states will replace the points $\vec{n}$ and $\vec{n}_0$ when we go to the noncommutative fuzzy sphere.

It is not difficult to see that the symplectic 2-form on $\mathbf{S}^2$ given by $\omega{\equiv}-\frac{l}{2}\epsilon_{abc}n_cdn_a{\wedge}dn_b=ldcos\theta{\wedge}d\phi$ where l is an undetremined non-zero real number and with θ and ϕ being the usual angle coordinates, can also be rewritten in the form $\omega=ild(Tr\sigma_3g(\sigma,t)^{-1}dg(\sigma,t))$ where t is a time variable and σ is a parameter in the range $[0,1]$. This means in particular that the quantization of the above symplectic 2-form ω is equivalent to the quantization of the Wess-Zumino term

$$L=ilTr\bigg(\sigma_3g^{-1}\dot{g}\bigg). \tag{2.2}$$

Indeed if we define a triangle Δ in the plane (t,σ) by its boundaries $\partial\Delta_1 = (\sigma, t_1)$, $\partial\Delta_2 = (\sigma, t_2)$ and $\partial\Delta_3 = (1, t)$ then it is a trivial exercise to show that

$$S_{WZ} = \int_{\Delta} \omega = \int_{t_1}^{t_2} Ldt + il \int_0^1 Tr\sigma_3 \Big(g(\sigma,t_1)^{-1}\partial_\sigma g(\sigma,t_1) - g(\sigma,t_2)^{-1}\partial_\sigma(\sigma,t_2) \Big). \quad (2.3)$$

It is a known result that the quantization of the above Wess-Zumino Lagrangian (2.2) will give all $SU(2)$ irreducible representations with spins $s \equiv l$, i.e the values of l in the quantum theory become strictly quantized .

The physical wave functions of the quantum system are complex valued functions on $SU(2)$ of the form

$$\psi(g) = \sum_{m=-l}^{l} C_m < lm|U^{(l)}(g)|ll >, \quad (2.4)$$

with scalar product defined by $(\psi_1, \psi_2) = \int_{SU(2)} d\mu(g)\psi_1(g)^*\psi_2(g)$ where $d\mu$ is the Haar measure on $SU(2)$. $U^{(l)}(g)$ is the IRR l of $g \in SU(2)$. Obviously $< lm|U^{(l)}(g)|ll >$ transforms as the heighest weight state (l,l) under the right action $g \longrightarrow gg_1$ of the group $g_1 \in SU(2)$, while under the left action $g \longrightarrow g_1 g$, $\{< lm|U^{(l)}(g)|ll >\}$ transforms as a basis of the Hilbert space $\mathbf{H}_l$ of the $(2l+1)-$dimensional IRR of $SU(2)$. This left action is clearly generated by the usual angular momenta of $SU(2)$ in the IRR l , namely

$$[L_a, L_a] = i\epsilon_{abc}L_c \ , \ \sum_{a=1}^{3} L_a^2 = l(l+1). \quad (2.5)$$

Explicitly this action is given by

$$[iL_a\psi](g) = \left[\frac{d}{dt}\psi\Big(e^{-i\frac{\sigma_a}{2}t}g\Big)\right]_{t=0}. \quad (2.6)$$

The algebra $\mathbf{A}$ of all observables of the system is the algebra of linear operators which act on the left of $\psi(g)$ by left translations , i.e an arbitrary linear operator $\phi^F \in \mathbf{A}$ will admit in general an expansion of the form

$$\phi^F = \sum_{a_1,..a_k} \alpha_{a_1..a_k} L_{a_1}...L_{a_k}. \quad (2.7)$$

The sum in (2.7) as we will check shortly is actually cut-off . The algebra $\mathbf{A}$ of all observables of the system is therefore a matrix algebra Mat_{2l+1} , while physical wave functions span a $(2l+1)-$dimensional Hilbert space $\mathbf{H}_l$ on which these observables are naturally acting [see [4] and references therein for more detail] .

We define the noncommutative fuzzy sphere by Connes spectral triple $(Mat_{2l+1}, \mathbf{H}_l, \Delta^F)$ [1] . The matrix algebra Mat_{2l+1} is the above algebra $\mathbf{A}$ of $(2l+1)\times(2l+1)$ matrices which acts on the $(2l+1)-$dimensional Hilbert space $\mathbf{H}_l$ of the IRR l of $SU(2)$. Matrix coordinates on $\mathbf{S}^2_F$ are defined by [2, 3, 4]

$$(n_1^F)^2 + (n_2^F)^2 + (n_3^F)^2 = 1 \ , \ [n_a^F, n_a^F] = \frac{i}{\sqrt{l(l+1)}}\epsilon_{abc}n_c^F, \quad (2.8)$$

with

$$n_a^F = \frac{L_a}{\sqrt{l(l+1)}}. \tag{2.9}$$

A "fuzzy" function on $\mathbf{S}_F^2$ is a linear operator $\phi^F \in Mat_{2l+1}$ which can also be defined by an expansion of the form (2.7) .

Derivations on $\mathbf{S}_F^2$ are on the other hand defined by the generators of the adjoint action of $SU(2)$, in other words the derivative of the fuzzy function ϕ^F in the space-time direction a is the commutator $[L_a, \phi^F]$. This can also be put in the form

$$AdL_a(\phi^F) \equiv [L_a, \phi^F] = (L_a^L - L_a^R)(\phi^F), \tag{2.10}$$

where L_a^L's and $-L_a^R$'s are the generators of the IRR l of $SU(2)$ which act respectively on the left and on the right of the algebra Mat_{2l+1} , i.e $L_a^L\phi^F \equiv L_a\phi^F$, $L_a^R\phi^F \equiv \phi^F L_a$ for any $\phi^F \in Mat_{2l+1}$. Hence the Laplacian operator Δ^F on the fuzzy sphere is simply given by the Casimir operator

$$\Delta^F = (L_a^L - L_a^R)^2. \tag{2.11}$$

The algebra of matrices Mat_{2l+1} decomposes therefore under the action of the group $SU(2)$ as $l \otimes l = 0 \oplus 1 \oplus 2 \oplus .. \oplus 2l$ [The first l stands for the left action of the group while the other l stands for the right action]. As a consequence a general scalar function on $\mathbf{S}_F^2$ can be expanded in terms of polarization tensors as follows

$$\phi^F = \sum_{k=0}^{2l} \sum_{m=-k}^{k} \phi_{km} \hat{Y}_{km}(l). \tag{2.12}$$

[For an extensive list of the properties of $\hat{Y}_{km}(l)$'s see [5]] . This expansion is equivalent to (2.7) but now the cut-off is made explicit . The fact that the summation over k involves only angular momenta which are $\leq 2l$ originates from the fact that the spectrum $k(k+1)$ of the Laplacian Δ^F is cut-off at $k = 2l$. As we will show in this article this rotationally-invariant cut-off is non-trivial in the sense that it respects both gauge and chiral symmetries.

As one can already notice all these definitions are in very close analogy with the case of continuum $\mathbf{S}^2$ where the algebra of functions $\mathcal{A}$ plays there the same role played here by the matrix algebra Mat_{2l+1} . In fact the continuum limit is defined by $l \longrightarrow \infty$ where the fuzzy coordinates n_a^F's approach the ordinary coordinates n_a's and where the algebra Mat_{2l+1} tends to the algebra $\mathcal{A}$ in the sense that

$$\phi^F \longrightarrow \phi(\vec{n}) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \phi_{km} Y_{km}(\vec{n}). \tag{2.13}$$

In above $Y_{km}(\vec{n})$ stands for the canonical spherical harmonics . Correspondingly fuzzy derivations reduce to ordinary ones on commutative $\mathbf{S}^2$, i.e $adL_a(\phi^F) \longrightarrow \mathcal{L}_a(\phi)(\vec{n})$, $\mathcal{L}_a = -i\epsilon_{abc} n_b \partial_c$. Formally one writes $\mathcal{A} = Mat_\infty$ and think of the fuzzy sphere as having a finite number of points equal to $2l+1$ which will diverge in the continuum limit $l \longrightarrow \infty$.

2.2 Star Product

To make the continuum limit more precise we will need to introduce the star product on $\mathbf{S}^2_F$. The irreducible representation l of $SU(2)$ can be obtained from the symmetric product of $2l$ fundamental representations $\frac{1}{2}$ of $SU(2)$. Given an element $g\in SU(2)$, its $l-$representation matrix $U^{(l)}(g)$ is given as follows

$$U^{(l)}(g)=U^{(\frac{1}{2})}(g)\otimes_s ...\otimes_s U^{(\frac{1}{2})}(g), 2l-times. \tag{2.14}$$

$U^{(\frac{1}{2})}(g)$ is the spin $\frac{1}{2}$ representation of $g\in SU(2)$. Clearly the states $|\vec{n}_0,\frac{1}{2}>$ and $|\vec{n},\frac{1}{2}>=g|\vec{n}_0,\frac{1}{2}>$ of $\mathbf{H}_{\frac{1}{2}}$ will correspond in $\mathbf{H}_l$ to the two states $|\vec{n}_0,l>$ and $|\vec{n},l>$ respectively such that

$$|\vec{n},l>=U^{(l)}(g)|\vec{n}_0,l> . \tag{2.15}$$

To any fuzzy scalar function ϕ^F on $\mathbf{S}^2_F$, i.e an operator ϕ^F acting on $\mathbf{H}_l$, we associate a "classical" function $<\phi^F>(\vec{n})$ on a classical $\mathbf{S}^2$ by

$$<\phi^F>(\vec{n})=<\vec{n},l|\phi^F|\vec{n},l>, \tag{2.16}$$

such that the product of two such operators ϕ^F_1 and ϕ^F_2 is mapped to the star product of the corresponding two functions

$$<\phi^F_1>*<\phi^F_2>(\vec{n})=<\vec{n},l|\phi^F_1\phi^F_2|\vec{n},l> . \tag{2.17}$$

A long calculation shows that this star product is given explicitly by [12]

$$\begin{aligned} <\phi^F_1>*<\phi^F_2>(\vec{n}) &= \sum_{k=0}^{2l}\frac{(2l-k)!}{k!(2l)!}K_{a_1b_1}....K_{a_kb_k}\frac{\partial}{\partial n^{a_1}}...\frac{\partial}{\partial n^{a_k}}<\phi^F_1>(\vec{n})\frac{\partial}{\partial n^{b_1}}...\frac{\partial}{\partial n^{b_k}}<\phi^F_2>(\vec{n}) \\ K_{ab} &= \delta_{ab}-n_an_b-i\epsilon_{abc}n_c. \end{aligned} \tag{2.18}$$

In these coherent states one can also compute

$$<n^F_a>=\frac{1}{\sqrt{1+\frac{1}{l}}}n_a\ ,\quad <[L_a,\phi^F]>=(\mathcal{L}_a<\phi^F>)(\vec{n}), \tag{2.19}$$

and

$$\frac{1}{2l+1}Tr_l\phi^F_1\phi^F_2=\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}<\phi^F_1>*<\phi^F_2>(\vec{n}). \tag{2.20}$$

The trace Tr_l is obviously taken over the Hilbert space $\mathbf{H}_l$. Remark finally that the coherent state $|\vec{n},l>$ becomes localized around the point $\vec{n}$ in the limit and as a consequence $Lim<n^F_a>=n_a$, $Lim<\phi^F>=\phi$ and $Lim<[L_a,\phi_a]>=\mathcal{L}_a\phi$, etc . In this limit the star product reduces also to the ordinary product of functions .

3. Fuzzy Actions

3.1 Gauge Fields on $\mathbf{S}^2_F$

Next we would like to write down the Schwinger model on $\mathbf{S}^2_F$. First we introudce 2−dimensional gauge fields on the fuzzy sphere and their action . A vector field $\vec{A}^F$ on the fuzzy sphere can be (similarly to scalar fields) defined by an expansion in terms of polarization tensors of the form

$$A^F_a=\sum_{k=0}^{2l}\sum_{m=-k}^{k}A_a(km)\hat{Y}_{km}(l)\ ,\ A^{F+}_a=A^F_a. \tag{3.1}$$

In the continuum limit this expansion reduces to $A_a(\vec{n})=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}A_a(km)Y_{km}(\vec{n})$. Each component A^F_a is a $(2l+1)\times(2l+1)$ matrix and the modes $A_a(km)$ are complex numbers satisfying $A_a(km)^*=(-1)^mA_a(k-m)$ where for each momentum (km) the corresponding triple $(A^F_1(km),A^F_2(km),A^F_3(km))$ transforms as an $SO(3)-$vector. We have also to note here that a much natural expansion of vector fields on the finite dimensional fuzzy sphere can be given instead in terms of "vector" polarization tensors [5]. The expansion (3.1) is however enough for the purpose of this article since we will mostly deal with the fermion action . Writing a gauge principle for this matrix vector field is not difficult, indeed the action takes the usual form

$$S_{YMF}=\frac{1}{4e^2}\frac{1}{2l+1}Tr_lF^F_{ab}F^F_{ab}. \tag{3.2}$$

The curvature $F^F_{ab}=-F^{F+}_{ab}$ is also given by the usual formula $F^F_{ab}{\equiv}[D^F_a,D^F_b]-i\epsilon_{abc}D^F_c$ with the covariant derivative $D^F_a=L_a+A^F_a$ or equivalently $F^F_{ab}=[L_a,A^F_b]-[L_b,A^F_a]+[A^F_a,A^F_b]-i\epsilon_{abc}A^F_c$. Gauge transformations are implemented by unitary transformations acting on the $(2l+1)-$dimensional Hilbert space of the irreducible representation l of $SU(2)$. These transformations are $D^F_a{\longrightarrow}D^{F'}_a=U^FD^F_aU^{F+}$, $A^F_a{\longrightarrow}A^{F'}_a=U^FA^F_aU^{F+}+U^F[L_a,U^{F+}]$ and $F^F_{ab}{\longrightarrow}F^{F'}_{ab}=U^FF^F_{ab}U^{F+}$ where $U^F=e^{i\Omega^F}$ and $\Omega^F=\Omega^{F+}$ is an element of the algebra Mat_{2l+1} of $(2l+1)\times(2l+1)$ matrices . U^F's define then fuzzy $U(1)$ gauge theory . Clearly $U(1)_F{\equiv}U(2l+1)$.

It is not difficult to convince ourselves that (3.2) has the correct continuum limit , namely

$$S_{YMF}{\longrightarrow}S_{YM}=\frac{1}{4e^2}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}F_{ab}F_{ab}\ ,\ l{\longrightarrow}\infty. \tag{3.3}$$

For example one can use the star product (2.18) on $\mathbf{S}^2_F$ to see that $A_a=Lim_{l\longrightarrow\infty}<A^F_a>$ and $F_{ab}=Lim_{l\longrightarrow\infty}<F^F_{ab}>=\mathcal{L}_aA_b-\mathcal{L}_bA_a-i\epsilon_{abc}A_c$. Also by using (2.20) it is seen that the trace $\frac{1}{2l+1}Tr_l$ behaves as the integral $\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}$ in the limit where the star product becomes the commutative product .

A final remark concerning vector fields is to note that the gauge field $\vec{A}^F$ has three components and hence an extra condition is needed in order to project this gauge field onto two dimensions . One adopts here the prescription of [11], i.e we impose on the gauge filed $\vec{A}^F$ the gauge-covariant condition

$$D_a^F D_a^F = l(l+1). \tag{3.4}$$

This constraint reads explicitely $\{n_a^F, A_a^F\} = -(\vec{A}^F)^2/\sqrt{l(l+1)}$, and thus it is not difficult to check that in the continuum limit $l \longrightarrow \infty$ the normal component of the gauge field is zero , i.e $\phi \equiv \vec{n}.\vec{A} = 0$.

3.2 Fermion Doubling

In close analogy with the free fermion action on ordinary $\mathbf{S}^2$, free fermion action on fuzzy $\mathbf{S}^2$ is defined by

$$S_F = \frac{1}{2l+1} Tr_l \bar{\psi}_F \mathcal{D}_F \psi_F \ , \ \mathcal{D}_F = \sigma_a [L_a, ...] + 1. \tag{3.5}$$

$\mathcal{D}_F$ is precisely the Dirac operator on fuzzy $\mathbf{S}^2$ [4, 7, 8, 9] , and σ_i's are Pauli matrices . The fuzzy spinor ψ_F is an element of $Mat_{2l+1} \otimes \mathbf{C}^2$, it is of mass dimension $(\text{mass})^{\frac{1}{2}}$ and is such that $\bar{\psi}_F = \psi_F^+$.

The so-called Grosse-Klimčík-Prešnajder Dirac operator $\mathcal{D}_F$ on fuzzy $\mathbf{S}^2$ admits a chirality operator which can be seen as follows , first we rewrite $\mathcal{D}_F$ in the form [7, 4, 8, 9]

$$\frac{1}{2l+1} \mathcal{D}_F = \frac{1}{2} (\Gamma^R + \Gamma^L), \tag{3.6}$$

where Γ^R and Γ^L are the operators

$$\Gamma^L = \frac{1}{l+\frac{1}{2}} [\vec{\sigma}.\vec{L}^L + \frac{1}{2}] \ , \ \Gamma^R = \frac{1}{l+\frac{1}{2}} [-\vec{\sigma}.\vec{L}^R + \frac{1}{2}]. \tag{3.7}$$

By analogy with (2.9) we also define $n_a^L = \frac{L_a^L}{\sqrt{l(l+1)}}$ and $n_a^R = \frac{L_a^R}{\sqrt{l(l+1)}}$ with obvious continuum limits , i.e $n_a^L, n_a^R \longrightarrow n_a$ when $l \longrightarrow \infty$ (since the left and right actions become identical in the limit). Remark that one can also make the identification $L_a^L \equiv L_a$, $n_a^L \equiv n_a^F$. In general all operators acting from the left can be thought of as elements of the algebra Mat_{2l+1}.

Γ^R is the chirality operator as was shown originally in [10] , this choice is also motivated by the fact that $\Gamma^R \longrightarrow -\gamma = -\vec{\sigma}\vec{n}$, when $l \longrightarrow \infty$, $(\Gamma^R)^2 = \Gamma^R$, $(\Gamma^R)^+ = \Gamma^R$, and $[\Gamma^R, \phi^F] = 0$ for any $\phi^F \in Mat_{2l+1}$. However this Γ^R does not exactly anticommute with the Dirac operator since

$$\mathcal{D}_F \Gamma^R + \Gamma^R \mathcal{D}_F = \frac{1}{l+\frac{1}{2}} \mathcal{D}_F^2. \tag{3.8}$$

The continuum limit of this equation is simply given by the canonical anticommutation relation $\{\mathcal{D}, \gamma\} = 0$ where the continuum Dirac operator $\mathcal{D}$ is given by $\mathcal{D} = \sigma.\mathcal{L} + 1$. Remark that for all practical purposes Γ^L is also a chirality operator , it has the correct continuum limit , i.e $\Gamma^L \longrightarrow \gamma$ when $l \longrightarrow \infty$, it satisfies $(\Gamma^L)^2 = 1$, $(\Gamma^L)^+ = \Gamma^L$, and by using (3.6) one rewrites (3.8) in the form

$$\mathcal{D}_F \Gamma^L + \Gamma^L \mathcal{D}_F = \frac{1}{l+\frac{1}{2}} \mathcal{D}_F^2. \tag{3.9}$$

Indeed Γ^L fails only to commute with the elements of the algebra Mat_{2l+1} . For later use we notice that (3.8) and (3.9) can also be rewritten in the form

$$\mathcal{D}_F \Gamma^R - \Gamma^L \mathcal{D}_F = 0. \tag{3.10}$$

It was shown in [4, 7] that the pair (Γ^R, Γ^L) defines a chiral structure on fuzzy $\mathbf{S}^2$ which satisfies $a)$ the Ginsparg-Wilson relation , $b)$ is without fermion doubling and $c)$ has the correct continuum limit . Indeed Γ^R and Γ^L together with the identity generate a Ginsparg-Wilson algebra where the canonical Dirac-Ginsparg-Wilson operator is defined by $\mathcal{D}_{DGW} = \Gamma^R \mathcal{D}_F$ while the lattice spacing is identified as $a = \frac{2}{2l+1}$ [see [19] and references therein for more detail] . The Ginsparg-Wilson relation for zero gauge field is essentially given by equation (3.10) .

On the other hand the absence of fermion doubling can be seen by comparing the spectrum of $\mathcal{D}_F$ which can be easily computed to be given by

$$\mathcal{D}_F(j) = \{\pm(j+\frac{1}{2}), j = \frac{1}{2}, \frac{3}{2}, ..., 2l - \frac{1}{2}\} \cup \{j + \frac{1}{2} \quad , \quad j = 2l + \frac{1}{2}\} \tag{3.11}$$

with the spectrum of the continuum Dirac operator $\mathcal{D}$ given by $\mathcal{D}(j) = \{\pm(j+\frac{1}{2}), j = \frac{1}{2}, \frac{3}{2}, ..., \infty\}$ [9] . As one can immediately notice there is no fermion doubling and the spectrum of $\mathcal{D}_F$ is simply cut-off at the top eigenvalue $j = 2l + \frac{1}{2}$. The continuum limit of this chiral structure is therefore obvious by construction .

The fuzzy gauged Dirac operator is obviously defined by $\mathcal{D}_{AF} = \mathcal{D}_F + \sigma_a A_a^F$ and thus the fuzzy gauged action is given by

$$S_{AF} = \frac{1}{2l+1} Tr_l \bar{\psi}_F \mathcal{D}_{AF} \psi_F \equiv \frac{1}{2l+1} Tr_l \bigg[\bar{\psi}_F \sigma_a [L_a, \psi_F] + \bar{\psi}_F \psi_F + \bar{\psi}_F \sigma_a A_a^F \psi_F \bigg]. \tag{3.12}$$

The spinor ψ_F is assumed here to transform in the fundamental representation of the fuzzy gauge group $U(1)_F \equiv U(2l+1)$, i.e $\psi_F \longrightarrow \psi_F^{'} = U^F \psi_F$, $\bar{\psi}_F \longrightarrow \bar{\psi}_F^{'} = \bar{\psi}_F U^{F+}$.

Again it is not difficult to see that both classical actions (3.5) and (3.12) behave correctly in the continuum limit in the sense that $S_F \longrightarrow S = \int_{\mathbf{S}^2} \frac{d\Omega}{4\pi} \bar{\psi} \mathcal{D} \psi$ and $S_{AF} \longrightarrow S_A = \int_{\mathbf{S}^2} \frac{d\Omega}{4\pi} \bar{\psi} \mathcal{D}_A \psi$ when $l \longrightarrow \infty$ and where $\psi = Lim_{l \longrightarrow \infty} < \psi^F >$ and $\mathcal{D}_A = \mathcal{D} + \sigma_a A_a$. Explicitly we write

$$S = \int_{\mathbf{S}^2} \frac{d\Omega}{4\pi} \bigg[\bar{\psi} \sigma_a \mathcal{L}_a(\psi) + \bar{\psi} \psi \bigg], \tag{3.13}$$

and

$$S_A = \int \frac{d\Omega}{4\pi} \bigg[\bar{\psi} \sigma_a \mathcal{L}_a(\psi) + \bar{\psi} \psi + \bar{\psi} \sigma_a A_a \psi \bigg]. \tag{3.14}$$

By putting the actions (3.2) and (3.12) together we obtain the fuzzy Schwinger model on $\mathbf{S}^2_F$, namely

$$S_{Schwinger}=\frac{1}{4e^2}\frac{1}{2l+1}Tr_lF^F_{ab}F^F_{ab}+\frac{1}{2l+1}Tr_l\bar{\psi}_F\mathcal{D}_{AF}\psi_F, \tag{3.15}$$

where the gauge field $\vec{A}^F$ is also assumed to satisfy the constraint (3.4) .

4. Quantum Chiral Symmetry on $\mathbf{S}^2_F$

4.1 Fermion Propagator

The quantum theory of interest is defined through the following path integral

$$\int\mathcal{D}A^F_i\int\mathcal{D}\psi_F\mathcal{D}\bar{\psi}_Fe^{-S_{Schwinger}}. \tag{4.1}$$

The anomaly arises generally from the non-invariance of the fermionic measure under chiral transformations [13] and hence we will focus first on this measure and show explicitly that for all finite approximations of the noncommutative Schwinger model on $\mathbf{S}^2_F$ this measure is in fact exactly invariant . Indeed we will show shortly that the $U(1)$ "fuzzy"axial anomaly on $\mathbf{S}^2_F$ comes entirely from the non-invariance of the action due to edge effect . In the appendix we will also discuss how one can shift the anomaly from the action back to the measure.

It is also enough to treat in the following the gauge field as a background field since all we want to compute is the fuzzy axial anomaly on $\mathbf{S}^2_F$ and its continuum limit the canonical local axial anomaly on $\mathbf{S}^2$, i.e only the fermion loop is relevant . Quantization of the Yang-Mills action S_{YMF} will be reported elsewhere .

In a matrix model such as (4.1) manipulations on the quantum measure have a precise meaning . Indeed and by following [13] we first expand the fuzzy spinors ψ_F , $\bar{\psi}_F$ in terms of the eigentensors $\phi(\mu,A)$ of the Dirac operator $\mathcal{D}_{AF}$, write $\psi_F=\sum_\mu\theta_\mu\phi(\mu,A)$, $\bar{\psi}_F=\sum_\mu\bar{\theta}_\mu\phi^+(\mu,A)$ where θ_μ's , $\bar{\theta}_\mu$'s are independent sets of Grassmanian variables, and $\phi(\mu,A)$'s are defined by $\mathcal{D}_{AF}\phi(\mu,A)=\lambda_\mu(A)\phi(\mu,A)$, and normalized such that

$$\frac{1}{2l+1}Tr_l\phi^+(\mu,A)\phi(\nu,A)=\delta_{\mu\nu}. \tag{4.2}$$

For zero fuzzy gauge fields μ stands for j , k and m which are the eigenvalues of $\vec{J}^2=(\vec{K}+\frac{\vec{\sigma}}{2})^2$, $\vec{K}^2=(\vec{L}^L-\vec{L}^R)^2$ and J_3 respectively . Indeed the asymptotic behaviour when $A^F_a\longrightarrow 0$ of $\phi(\mu,A)$'s and $\lambda_\mu(A)$'s is given by $\lambda_\mu(A)\longrightarrow j(j+1)-k(k+1)+\frac{1}{4}$ and $\phi(\mu,A)\longrightarrow\sqrt{2l+1}\sum_{k_3,\sigma}C^{jm}_{kk_3\frac{1}{2}\sigma}\hat{Y}_{kk_3}(l)\chi_{\frac{1}{2}\sigma}$ [4, 5, 9] . The quantum measure is therefore well defined and it is given by

$$\mathcal{D}\psi_F\mathcal{D}\bar{\psi}_F=\prod_\mu d\theta_\mu d\bar{\theta}_\mu\longrightarrow\prod_{k=0}^{2l}\prod_{j=k-\frac{1}{2}}^{k+\frac{1}{2}}\prod_{m=-j}^{j}d\theta_{kjm}d\bar{\theta}_{kjm} \tag{4.3}$$

From the action S_{AF} and the identity (4.2) one can compute the propagator $<\theta_\mu\bar{\theta}_\nu>=\delta_{\mu\nu}/\lambda_\mu(A)$ or equivalently

$$<\psi^{AB}_{F\alpha}\bar{\psi}^{CD}_{F\beta}>_{ev}=\sum_\mu\frac{1}{\lambda_\mu(A)}\phi^{AB}_\alpha(\mu,A)\phi^{+CD}_\beta(\mu,A)\equiv(2l+1)\bigg(\frac{1}{\mathcal{D}_{AF}}\bigg)^{AB,DC}_{\alpha\beta}. \quad (4.4)$$

$<>_{ev}$ stands for expectation value . We are assuming no monopole configurations and thus the inverse of the gauged Dirac operator is always well defined . This inverse is defined by the formula

$$(\mathcal{D}_{AF})^{C^{'}D^{'},AB}_{\gamma\alpha}(\frac{1}{\mathcal{D}_{AF}})^{AB,DC}_{\alpha\beta}=\delta_{\gamma\beta}\delta^{C^{'}D}\delta^{D^{'}C}. \quad (4.5)$$

In above we have also used the fact that because the Dirac operator $\mathcal{D}_{AF}$ is self-adjoint on $Mat_{2l+1}{\otimes}\mathbf{C}^2$, the states $\phi(\mu,A)$'s must form a complete set , viz

$$\frac{1}{2l+1}\sum_\mu\phi^{AB}_\alpha(\mu,A)\phi^{+CD}_\beta(\mu,A)=\delta_{\alpha\beta}\delta^{AD}\delta^{BC}. \quad (4.6)$$

The Dirac operator $(\mathcal{D}_{AF})_{\alpha\beta}$ and the propagator $(\mathcal{D}^{-1}_{AF})_{\alpha\beta}$ carry 4 indices because they can act on matrices of Mat_{2l+1} either from the left or from the right , for example $\mathcal{D}_{AF}$ acts explicitly as $(\mathcal{D}_{AF}\phi(A,\mu))^{AB}_\alpha=(\mathcal{D}_{AF})^{AB,CD}_{\alpha\beta}\phi^{CD}_\beta$ where

$$(\mathcal{D}_{AF})^{AB,CD}_{\alpha\beta}=(\sigma_a)_{\alpha\beta}(D^F_a)^{AC}\delta^{BD}-(\sigma_a)_{\alpha\beta}(L_a)^{DB}\delta^{AC}+\delta_{\alpha\beta}\delta^{AC}\delta^{BD}. \quad (4.7)$$

4.2 The Dirac-Ginsparg-Wilson Relation on $\mathbf{S}^2_F$

We will now undertake the task of deriving the Dirac-Ginsparg-Wilson relation on $\mathbf{S}^2_F$ in the presence of a gauge field . As it turns out this relation contains exactly a fuzzy anomaly which will become a local anomaly in the limit . We first start with the free theory and rewrite the Ginsparg-Wison relation (3.10) in the form $(\Gamma^R{-}\Gamma^L)\mathcal{D}_F{+}\mathcal{D}_F(\Gamma^R{-}\Gamma^L)=0$.This means that in the absence of gauge fields we must have $tr[\Gamma^R-\Gamma^L]=0$ where the trace is taken in the space of spinors , in other words the anomaly vanishes. However if we include the gauge field through the gauge-covariant Dirac operator $\mathcal{D}_{AF}$ which can also be written in the form

$$\frac{1}{2l+1}\mathcal{D}_{AF}=\frac{1}{2}(\Gamma^R+\hat{\Gamma}^L), \quad (4.8)$$

with

$$\hat{\Gamma}^L=\frac{1}{l+\frac{1}{2}}\bigg[\sigma_aD^F_a+\frac{1}{2}\bigg]=\Gamma^L+\frac{1}{l+\frac{1}{2}}\vec{\sigma}.\vec{A}^F. \quad (4.9)$$

[(4.8) is to be compared with the free formula (3.6)] . Then we can compute instead the gauge-covariant anticommutation relation

$$\{\Gamma^R-\hat{\Gamma}^L,\mathcal{D}_{AF}\}=-\frac{4}{2l+1}\bigg(F^F+(D^F_a)^2-l(l+1)\bigg)$$
$$F^F=\frac{i}{2}\epsilon_{abc}\sigma_cF^F_{ab}. \quad (4.10)$$

The continuum limit of this equation is $\{\gamma, \mathcal{D}_A\} = 2\phi = 0$ where ϕ is the normal component of the gauge field on $\mathbf{S}^2$ which is zero by the continuum limit of (3.4). In other words , in the continuum interacting theory one might be tempted to conclude that $tr\gamma = 0$ which we know is wrong in the presence of gauge fields. Noncommutative geometry , as it is already obvious from equation (4.10), already gives us the structure of the chiral anomaly, indeed by using the constraint (3.4) one can put (4.10) in the equivalent form

$$\mathcal{D}_{AF}\Gamma^R - \hat{\Gamma}^L\mathcal{D}_{AF} = -\frac{2}{2l+1}F^F = -\frac{i}{2l+1}\epsilon_{abc}\sigma_c F^F_{ab}. \tag{4.11}$$

We recognize immediately the left hand side as a fuzzy anomaly since it vanishes in the limit $A^F \longrightarrow 0$ where (4.11) reduces to (3.10) . We will now show that this corresponds indeed to the actual global $U_A(1)$ fuzzy anomaly on $\mathbf{S}^2_F$.

In the following we will also need to write down the explicit action of the operators Γ^R , $\hat{\Gamma}^L$. We have

$$\begin{aligned}
(\Gamma^R)^{AB,CD}_{\alpha\beta} &= \frac{\delta^{AC}}{l+\frac{1}{2}}\bigg(-(\sigma_a)_{\alpha\beta}L_a^{DB} + \frac{1}{2}\delta_{\alpha\beta}\delta^{DB}\bigg) \\
(\hat{\Gamma}^L)^{AB,CD}_{\alpha\beta} &= \frac{\delta^{BD}}{l+\frac{1}{2}}\bigg((\sigma_a)_{\alpha\beta}(D_a^F)^{AC} + \frac{1}{2}\delta_{\alpha\beta}\delta^{AC}\bigg) \equiv \delta^{BD}(\hat{\Gamma}^L)^{AC}_{\alpha\beta}.
\end{aligned} \tag{4.12}$$

In particular from the second equation we note that the operator $(\hat{\Gamma}^L)_{\alpha\beta}$ since it acts from the left it can also be thought of as an element of the algebra Mat_{2l+1}.

4.3 Gauge-Covariant Axial Current

Now we need to define chiral symmetry on the fuzzy sphere which must also be consistent with gauge invariance. In the absence of gauge fields and motivated by (3.10) we define chiral transformations by

$$\psi_F \longrightarrow \psi^{'}_F = \psi_F + (\Gamma^R\lambda^F\psi_F) + O((\lambda^F)^2)\ ,\ \bar{\psi}_F \longrightarrow \bar{\psi}^{'}_F = \bar{\psi}_F - (\bar{\psi}_F\lambda^F\Gamma^L) + O((\lambda^F)^2). \tag{4.13}$$

It is then not difficult to check that the action S_F changes by a total divergence. The chiral parameter λ^F is an arbitrary matrix in Mat_{2l+1} . In the presence of gauge fields it is therefore obvious that the minimal prescription is given by

$$\psi_F \longrightarrow \psi^{'}_F = \psi_F + (\Gamma^R\lambda^F\psi_F) + O((\lambda^F)^2)\ ,\ \bar{\psi}_F \longrightarrow \bar{\psi}^{'}_F = \bar{\psi}_F - (\bar{\psi}_F\lambda^F\hat{\Gamma}^L) + O((\lambda^F)^2), \tag{4.14}$$

where $\hat{\Gamma}^L$ is the $A-$dependent chirality-like operator defined in (4.9) . These transformations are of course motivated by (4.11) . Indeed one can check that (4.14) reduces in the limit to the usual chiral transformations yet it guarantees gauge invariance in the noncommutative fuzzy setting since both $\hat{\Gamma}^L$ and the chiral parameter λ^F transform covariantly as $U^F\hat{\Gamma}^L U^{F+}$ and $U^F\lambda^F U^{F+}$ respectively under gauge transformations . Naive fuzzy chiral transformations which would be again given by (4.14) but with Γ^L instead of $\hat{\Gamma}^L$ are in fact inconsistent with gauge symmetry as one might easily convince ourselves , whereas in the case of (4.14) if one gauge transform ψ_F and $\bar{\psi}_F$ by a unitary transformation U^F their chiral transform $\psi_F^{'}$ and $\bar{\psi}_F^{'}$ will also rotate by the same gauge transformation U^F. For completeness we write the meaning of (4.14) explicitly as follows

$$\psi_{F\alpha}^{'} = \psi_{F\alpha} + \frac{1}{2l+1}\bigg(- 2(\sigma_a)_{\alpha\beta}\lambda^F\psi_{F\beta}L_a + \lambda^F\psi_{F\alpha}\bigg) + O((\lambda^F)^2)$$
$$\bar{\psi}_{F\alpha}^{'} = \bar{\psi}_{F\alpha} - \frac{1}{2l+1}\bigg(2\bar{\psi}_{F\beta}\lambda^F(\sigma_a)_{\beta\alpha}D_a^F + \bar{\psi}_{F\alpha}\lambda^F\bigg) + O((\lambda^F)^2). \qquad (4.15)$$

The change of the action under these fuzzy chiral transformations (4.14) is given by

$$\int \mathcal{D}\psi_F{}^{'}\mathcal{D}\bar{\psi}_F^{'}e^{-S_{AF}^{'}} = \int \mathcal{D}\psi_F\mathcal{D}\bar{\psi}_F e^{S_{\theta F}}e^{-S_{AF}-\Delta S_{AF}}, \qquad (4.16)$$

with

$$\Delta S_{AF} = -\frac{1}{2l+1}Tr_l\lambda^F[D_a^F,\mathcal{J}_a^5] - \frac{i}{(2l+1)^2}\epsilon_{abc}Tr_l\bar{\psi}_F\lambda^F\sigma_cF_{ab}^F\psi_F$$
$$\mathcal{J}_a^5 = \bar{\psi}_F\sigma_a\Gamma^R\psi_F - [(\bar{\psi}_F\sigma_a)_\alpha,(\Gamma^R\psi_F)_\alpha]. \qquad (4.17)$$

An immediate remark is that the change in the action is not simply a total covariant divergence but there is an extra piece which vanishes (as it stands) only in the continuum limit, i.e we have a gauge-invariant edge effect. It is clear that the source of this edge effect is the RHS of the Ginsparg-Wilson relation (4.11).

The theta term in the path integral (4.16) is also gauge-invariant and it is given explicitly by

$$S_{\theta F} = -\frac{1}{2l+1}Tr_l\sum_\mu \phi^+(\mu,A)\lambda^F(\Gamma^R - \hat{\Gamma}^L)\phi(\mu,A). \qquad (4.18)$$

[This is gauge-invariant since for example $\phi(\mu,A)$ must transform as $U^L\phi(\mu,A)$ in order for the Dirac equation to be gauge-invariant]. Due to the finiteness of the matrix model , it is an identity easy to check that the theta term $S_{\theta F}$ is zero , indeed by using (4.6) and (4.12) we compute

$$S_{\theta F} = 2Tr_l\bigg(\lambda^F tr_2(\sigma D^F)\bigg) + \frac{2}{2l+1}Tr_l(\lambda^F)Tr_l\bigg(tr_2(\sigma L)\bigg)\equiv 0. \qquad (4.19)$$

tr_2 is the 2−dimensional spin trace , i.e $tr_2\mathbf{1} = 2$, $tr_2\sigma_a = 0$, etc . In fact it is this trace which actually vanishes, and hence it seems that the chiral WT identity $\Delta S_{GF} = S_{\theta F}$ is not anomalous due to the non-invariance of the measure under chiral transformations but rather anomalous due to edge effects (the second term of the first equation of (4.17)) . We write this WT identity $\Delta S_{GF} = 0$ in the form

$$<[D_a^F,\mathcal{J}_a^5]>_{ev} = \frac{i}{2l+1}\epsilon_{abc}<(\sigma_c)_{\alpha\beta}F_{ab}^F\psi_{F\beta}\bar{\psi}_{F\alpha}>_{ev}. \qquad (4.20)$$

Using the propagator (4.4) we find the anomaly

$$[D_a^F,<\mathcal{J}_a^5>]^{CB} = i\epsilon_{abc}(F_{ab}^F)^{CD}tr_2\sigma_c(D_{AF}^{-1})^{DA,BA}. \qquad (4.21)$$

This is the “fuzzy” form of the global $U_A(1)$ axial anomaly on $\mathbf{S}^2_F$. The inetgrated form of this anomaly is clearly given by

$$\mathcal{A}_{\theta F} = -\frac{i}{2l+1}\epsilon_{abc}(\lambda^F)^{BC}(F^F_{ab})^{CD}tr_2\sigma_c(\mathcal{D}^{-1}_{AF})^{DA,BA}. \tag{4.22}$$

The central claim of this article is that higher modes of the fuzzy sphere are essentially the source of the anomaly. In fact these top modes are the source of the edge effects we saw in (4.11) and in (4.17) which effectively yielded the non-vanishing anomaly (4.21) . Remark that if we make the top modes larger , i.e $l{\longrightarrow}\infty$, these effects in (4.11) and (4.17) become smaller while the fuzzy anomaly (4.21) remains non-zero . In the strict limit $l{\longrightarrow}\infty$ these edge effects in (4.11) and (4.17) completely disappear while the anomaly (4.21) survives [see below for the explicit proof]. This is the origin of the anomaly in this context .

5. The Continuum Limit

5.1 Gauge Covariant Expansion on $\mathbf{S}^2_F$

By definition the local axial anomaly on $\mathbf{S}^2$ is the continuum limit $l{\longrightarrow}\infty$ of the fuzzy axial anomaly (4.21) given also in (4.22). From equation (4.22) one can immediately notice that the computation (perturbative or otherwise) of the exact Dirac propagator $(\mathcal{D}_{AF})^{-1}$ is needed in order to obtain a closed formula of the fuzzy anomaly and its continuum limit. Towards this end the best approach is as usual to expand the above propagator in a gauge covariant manner and then calculate the anomaly. As it turns out there exists a gauge covariant expansion on the noncommutative fuzzy sphere which is not available in the continuum setting and which yields a non-perturbative result in the limit . In fact it is precisely in this sense that the fuzzy sphere is said to be a gauge-invariant, chiral-invariant regularization of the continuum physics . This is also in contrast with other approximation schemes in which gauge covariant expansions are so often absent .

This gauge covariant expansion can be motivated as follows . Starting with zero gauge field one can see that the free Dirac propagator $\frac{1}{\mathcal{D}_F}$ admits the expansion

$$\frac{1}{\mathcal{D}_F} = \frac{1}{a}\frac{1}{\mathcal{D}_F^2}(\Gamma^R+\Gamma^L) = \frac{a\Gamma^L}{2} + \frac{1}{\mathcal{D}_a}, \tag{5.1}$$

with

$$\frac{1}{\mathcal{D}_a} = ib\frac{1}{|\mathcal{D}_F|^2}\mathcal{D}^{'}_F\Gamma^L. \tag{5.2}$$

In above we have used the result $\Gamma^R\Gamma^L = -1+\frac{a^2}{2}(\mathcal{D}_F^2+2i\frac{b}{a}\mathcal{D}^{'}_F)$ as well as equation (3.10) where a is the lattice spacing introduced before , i.e $a=\frac{2}{2l+1}$, and $b=\sqrt{1-\frac{a^2}{4}}$.

$\mathcal{D}^{'}_F$ is the Watamura Dirac operator given by $\mathcal{D}^{'}_F = \epsilon_{abc}\sigma_a n^F_b L^R_c$ [4, 10]. The continuum limit $\mathcal{D}^{'}$ of $\mathcal{D}^{'}_F$ is related to $\mathcal{D}$ (the continuum limit of $\mathcal{D}_F$) by $\mathcal{D}^{'}=i\gamma\mathcal{D}$ and hence both operators $\mathcal{D}^{'}$ and $\mathcal{D}$ have the same spectrum . This does not mean that they commute since in fact we have $\{\mathcal{D}^{'},\mathcal{D}\}=0$. In the fuzzy , the spectrum of $(\mathcal{D}_F)^2$ as we have seen is simply cut-off at the top modes $j = 2l + \frac{1}{2}$ and is given by $(j + \frac{1}{2})^2$ while the spectrum of $(\mathcal{D}'_F)^2$ is deformed given by

$$(\mathcal{D}'_F(j))^2 = \left\{(j + \frac{1}{2})^2[1 + \frac{1 - (j + \frac{1}{2})^2}{4l(l+1)}] \ , \ j = \frac{1}{2}, \frac{3}{2}, ..., 2l + \frac{1}{2}\right\}. \tag{5.3}$$

In particular the eigenvalues of $\mathcal{D}'_F$ when $j = 2l + \frac{1}{2}$ are now exactly zero while for other large j's these eigenvalues are very small [10]. As a consequence of this behaviour we have in fact the exact anticommutation relation $\{\Gamma^R, \mathcal{D}'_F\} = 0$.

Next we remark that the "Dirac" operator $\mathcal{D}_a$ defined in (5.2) is such that $\mathcal{D}_a^{-2} = \mathcal{D}_F^{-2} - \frac{a^2}{4}$. In other words on the top modes $j = 2l + \frac{1}{2}$, $(\mathcal{D}_a)^2$ is strictly infinite whereas for the other large j's the eigenvalues of $(\mathcal{D}_a)^2$ are quite large. On the IR modes the spectrum of $(\mathcal{D}_a)^2$ is essentially equal to that of $\mathcal{D}_F^2$. Indeed one can explicitly compute the spectrum of $(\mathcal{D}_a)^2$ and one finds the result

$$(\mathcal{D}_a(j))^2 = \left\{(j + \frac{1}{2})^2 \frac{(2l+1)^2}{(2l + \frac{1}{2} - j)(2l + \frac{3}{2} + j)} \ , \ j = \frac{1}{2}, \frac{3}{2}, ..., 2l + \frac{1}{2}\right\}. \tag{5.4}$$

This means in particular that as far as the second term in (5.1) is concerned modes with large j's do not effectively propagate . We will show (after we also take the gauge field into account) that the contribution of these UV modes to the anomaly translates in the continuum limit into the contribution of contact terms whereas the contribution of the IR modes vanishes identically . As it turns out contact terms yield only a toplogical Chern-Simons Lagrangian which in two dimensions vanishes identically . This also reflects in a sense the fact that the above UV modes are suppressed in the propagator $(\mathcal{D}_a)^{-1}$. On the other hand although the first term $\frac{a\Gamma^L}{2}$ in (5.1) vanishes as it stands in the continuum limit , its contribution in that limit is not zero and is exactly given by the canonical theta term. Putting these facts together we conclude that the anomaly emerges essentially from the UV region of the spectrum as expected. We now give a rigorous proof of this result.

In the presence of gauge fields the free expansion (5.1) becomes a covariant expansion given by

$$\frac{1}{\mathcal{D}_{AF}} = \frac{1}{\mathcal{D}_{GF}^2} \frac{1}{a}\left(\frac{1}{2}\{\Gamma^R, \hat{\Gamma}^L\}\frac{1}{\hat{\Gamma}^{L2}} + \frac{1}{2}[\Gamma^R, \hat{\Gamma}^L]\frac{1}{\hat{\Gamma}^{L2}} + 1\right)\hat{\Gamma}^L. \tag{5.5}$$

We use now the results

$$\{\Gamma^R, \hat{\Gamma}^L\} = a^2\left[\mathcal{D}_{AF}^2 - F^F - \frac{2}{a^2}\right] \ , \ [\Gamma^R, \hat{\Gamma}^L] = 2iab\mathcal{D}'_{GF}. \tag{5.6}$$

F^F is defined in (4.10) and $\mathcal{D}'_{AF}$ is the gauged Watamuras Dirac operator defined by $\mathcal{D}'_{AF} = \epsilon_{abc}\sigma_a x_b^F L_c^R$ where x_b^F's are the covariant coordinates given by $x_b^F = \frac{D_b^F}{\sqrt{l(l+1)}}$. In other words x_b^F reduces to n_b^F in the absence of gauge fields and to n_b in the continuum limit. We use also the fact that $\hat{\Gamma}^L = b\sigma_a x_a^F + \frac{a}{2}$ to deduce the result $\hat{\Gamma}^{L2} = 1 + a^2 F^F$, then by putting all these ingredients together we find that the gauge covariant expansion of the full propagator is given by

$$\frac{1}{\mathcal{D}_{AF}} = \frac{a\hat{\Gamma}^L}{2} - \frac{a^3}{2}\frac{F^F}{1 + a^2F^F}\hat{\Gamma}^L + \frac{1}{\mathcal{D}_{Aa}}, \tag{5.7}$$

with

$$\frac{1}{\mathcal{D}_{Aa}}=\frac{1}{\mathcal{D}^2_{AF}}\left[ib\mathcal{D}^{'}_{AF}+\frac{a}{2}F^F\right]\frac{1}{1+a^2F^F}\hat{\Gamma}^L. \tag{5.8}$$

It is not difficult to see that each term in this expansion is exactly covariant under gauge transformations . It is also obvious that (5.7) reduces in the limit $A^F_a{\longrightarrow}0$ to (5.1) . Furthermore in the continuum limit $l{\longrightarrow}\infty$, (5.7) reduces to $\frac{1}{\mathcal{D}_A}=\frac{1}{\mathcal{D}^2_A}(i\mathcal{D}^{'}_A\gamma)$ which is actually an identity since $\mathcal{D}_A=i\mathcal{D}^{'}_A\gamma$ and thus the expansion (5.7) is simply not available to us in the continuum .

In order to compute the contribution of $(\mathcal{D}_{Aa})^{-1}$ it is obvious that one needs also an expansion of $\frac{1}{\mathcal{D}^2_{AF}}$. To this end we recall first that the square of the Dirac operator $\mathcal{D}_{AF}$ is given by $\mathcal{D}^2_{AF}=\mathcal{D}_{AF}+(L_a-L^R_a+A^F_a)^2+\frac{i}{2}\epsilon_{abc}\sigma_cF^F_{ab}$. By using the constraint (3.4) we can write this square in the form

$$\mathcal{D}^2_{AF}=2l(l+1)P_{AF}\left[1-\frac{1}{P_{AF}}v^F_an^R_a\right], \tag{5.9}$$

where n^R_a's are the fuzzy coordinates which act from the right , i.e $n^R_a=\frac{L^R_a}{\sqrt{l(l+1)}}$ and where

$$P_{AF}=1+\frac{1+F^F+\sqrt{l(l+1)}\sigma x^F}{2l(l+1)}\ ,\ v^F_a=x^F_a+\frac{\sigma_a}{2\sqrt{l(l+1)}}. \tag{5.10}$$

It is clear that the only bit in $\mathcal{D}^2_{AF}$ which acts on the right is $\vec{n}^R$ and that P_{AF} acts entirely on the left . This means that the operator $(P_{AF})_{\alpha\beta}$ can now be treated as a matrix in Mat_{2l+1} . We think now of P_{AF} as a propagator and of $n^R_av^F_a$ as a vertex and write the expansion

$$\begin{aligned}\frac{1}{\mathcal{D}^2_{AF}}&=\frac{1}{2l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v^F_an^R_a\right)^N\frac{1}{P_{AF}}\\&=\frac{1}{2l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v^F_{a_1}\frac{1}{P_{AF}}v^F_{a_2}...\frac{1}{P_{AF}}v^F_{a_N}\frac{1}{P_{AF}}\right)\left(n^R_{a_1}n^R_{a_2}...n^R_{a_N}\right).\end{aligned} \tag{5.11}$$

The meaning of this expansion will only be clear in the continuum limit which we will take shortly. The propagator $\frac{1}{P_{AF}}$ acts now from the left and thus it can also be thought of as a matrix in $Mat_{2l+1}{\otimes}Mat_2$ rather than as an operator . It is given by

$$\frac{1}{P_{AF}}=(1-\frac{a}{2}\hat{\Gamma}^L-\frac{a^2}{2}F^F)\frac{1}{1-\frac{a^3}{4-a^2}\{\hat{\Gamma}^L,F^F\}-\frac{a^4}{4-a^2}(F^F)^2}, \tag{5.12}$$

Therefore given any two operators $(X^L_a)_{\alpha\beta}$ and Y^R_b which act from the left and from the right respectively , i.e $(X^L_a)_{\alpha\beta}f{\equiv}(X_a)_{\alpha\beta}f$ and $Y^R_bf{\equiv}fY_b$ for any $f{\in}Mat_{2l+1}$, we can compute

$$\left(\frac{1}{\mathcal{D}_{AF}^2}X_a^LY_b^R\right)_{\alpha\beta}^{AB,CD}=\frac{1}{2l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F...\frac{1}{P_{AF}}v_{a_N}^F\frac{1}{P_{AF}}X_a\right)_{\alpha\beta}^{AC}\left(Y_bn_{a_N}^F...n_{a_2}^Fn_{a_1}^F\right)^{DB}. \tag{5.13}$$

To summarize , the gauge covariant expansion of the propagator $\frac{1}{\mathcal{D}_{AF}}$ is defined by equations (5.7) , (5.8), (5.11) and (5.12) . Using these equations together with the definition (5.13) one obtains an explicit formula for the exact quark propagator on $\mathbf{S}_F^2$, viz

$$\left(\frac{1}{\mathcal{D}_{AF}}\right)_{\alpha\beta}^{AB,CD}=\left(\frac{a\hat{\Gamma}^L}{2}-\frac{a^3}{2}\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right)_{\alpha\beta}^{AC}\delta^{BD}+\left(\frac{1}{\mathcal{D}_{Aa}}\right)_{\alpha\beta}^{AB,CD}, \tag{5.14}$$

with

$$\begin{aligned}\left(\frac{1}{\mathcal{D}_{Aa}}\right)_{\alpha\beta}^{AB,CD}&=\frac{ib\epsilon_{abc}}{2l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F...\frac{1}{P_{AF}}v_{a_N}^F\frac{1}{P_{AF}}\sigma_ax_b^F\frac{1}{1+a^2F^F}\hat{\Gamma}^L\right)_{\alpha\beta}^{AC}\left(L_cn_{a_N}^F...n_{a_2}^Fn_{a_1}^F\right)^{DB}\\&+\frac{a}{4l(l+1)}\sum_{N=0}^{\infty}\left(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F...\frac{1}{P_{AF}}v_{a_N}^F\frac{1}{P_{AF}}\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right)_{\alpha\beta}^{AC}\left(n_{a_N}^F...n_{a_2}^Fn_{a_1}^F\right)^{DB}.\end{aligned} \tag{5.15}$$

As we have explained this gauge covariant expansion does not exist on the continuum sphere . It will be used here to derive the fuzzy axial anomaly and its continuum limit the local axial anomaly .

5.2 Local Axial Anomaly

This expansion isolates in fact (in a gauge covariant fashion) the anomalous bit in the fermion propagator . More precisely , although the first term in (5.14) vanishes as it stands in the large l limit , its contribution (i.e its trace) gives exactly the canonical anomaly . Indeed we can easily compute

$$\begin{aligned}\delta_1\mathcal{A}_{\theta F}&\equiv-\frac{i}{2l+1}\epsilon_{abc}(\lambda^FF_{ab}^F)^{BD}\left[tr_2\sigma_c\left(\frac{a\hat{\Gamma}^L}{2}\right)^{DA,BA}+tr_2\sigma_c\left(-\frac{a^3}{2}\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right)^{DA,BA}\right]\\&=-2b\frac{i\epsilon_{abc}}{2l+1}Tr_l\lambda^FF_{ab}^F\delta_1x_c^F,\end{aligned} \tag{5.16}$$

where

$$\delta_1x_c^F=x_c^F-\frac{a^2}{2b}tr_2\left(\sigma_c\frac{F^F}{1+a^2F^F}\hat{\Gamma}^L\right). \tag{5.17}$$

This gauge-covariant vector $\vec{\delta}_1x^F$ in the continuum limit becomes exactly the unit vector $\vec{n}$ on the 2−dimensional sphere and hence the fuzzy anomaly (5.16) reduces in that limit to the theta term on $\mathbf{S}^2$, in other words

$$-2b\frac{i\epsilon_{abc}}{2l+1}Tr_l\lambda^FF_{ab}^F\delta_1x_c^F\longrightarrow-2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})n_c\ ,\ \text{when } l\longrightarrow\infty. \tag{5.18}$$

For example by using the star product on $\mathbf{S}^2_F$ one can see that in the continuum limit $l{\longrightarrow}\infty$ where $a{\longrightarrow}0$ and $b{\longrightarrow}1$ we have $\delta_1 x^F_a{\longrightarrow}n_a$, $F^F_{ab}{\longrightarrow}F_{ab}$, $\lambda^F{\longrightarrow}\lambda$ while the trace $\frac{1}{2l+1}Tr_l$ becomes the integral $\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}$.

5.3 The Chern-Simons Action

From equation (5.8) one can immediately read the remaining two extra corrections corresponding to the propagator $\frac{1}{\mathcal{D}_{Aa}}$, viz

$$\begin{aligned}\delta_2\mathcal{A}_{\theta F} &= -\frac{i}{2l+1}\epsilon_{abc}(\lambda^F F^F_{ab})^{BD} tr_2\sigma_c\bigg(\frac{1}{\mathcal{D}^2_{AF}}(ib\mathcal{D}^{'}_{AF})\frac{1}{1+a^2F^F}\hat{\Gamma}^L\bigg)^{DA,BA}\\ &= -i\epsilon_{abc}\bigg(\frac{1}{1+a^2F^F}\hat{\Gamma}^L\lambda^F F^F_{ab}\sigma_c\bigg)^{CD}_{\beta\alpha}\bigg(\frac{1}{\mathcal{D}^2_{AF}}(\frac{i}{2}b^2\epsilon_{pqr}\sigma_p x^F_q n^R_r)\bigg)^{DA,CA}_{\alpha\beta}, \quad (5.19)\end{aligned}$$

and

$$\begin{aligned}\delta_3\mathcal{A}_{\theta F} &= -\frac{i}{2l+1}\epsilon_{abc}(\lambda^F F^F_{ab})^{BD} tr_2\sigma_c\bigg(\frac{1}{\mathcal{D}^2_{AF}}(\frac{a}{2}F^F)\frac{1}{1+a^2F^F}\hat{\Gamma}^L\bigg)^{DA,BA}\\ &= -i\epsilon_{abc}\bigg(\frac{1}{1+a^2F^F}\hat{\Gamma}^L\lambda^F F^F_{ab}\sigma_c\bigg)^{CD}_{\beta\alpha}\bigg(\frac{1}{\mathcal{D}^2_{AF}}(\frac{a^2}{4}F^F)\bigg)^{DA,CA}_{\alpha\beta}. \quad (5.20)\end{aligned}$$

By using the expansion (5.11) with (5.12) together with the definition (5.13) we obtain for $\delta_2\mathcal{A}_{\theta F}$ and $\delta_3\mathcal{A}_{\theta F}$ the formulae

$$\begin{aligned}\delta_2\mathcal{A}_{\theta F} &= -2b\frac{i\epsilon_{abc}}{2l+1}Tr_l\lambda^F F^F_{ab}\delta_2 x^F_c\\ \delta_2 x^F_c &= \sum_{N=1}\bigg(\frac{1}{2l+1}Tr_l(n^F_{a_N}...n^F_{a_1}n^F_r)\bigg)\bigg(tr_2(\frac{1}{P_{AF}}v^F_{a_1}\frac{1}{P_{AF}}v^F_{a_2}...\frac{1}{P_{AF}}v^F_{a_N}\delta^F_{rc})\bigg) \quad (5.21)\end{aligned}$$

and

$$\begin{aligned}\delta_3\mathcal{A}_{\theta F} &= -2b\frac{i\epsilon_{abc}}{2l+1}Tr_l\lambda^F F^F_{ab}\delta_3 x^F_c\\ \delta_3 x^F_c &= \sum_{N=0}\bigg(\frac{1}{2l+1}Tr_l(n^F_{a_N}...n^F_{a_1})\bigg)\bigg(tr_2(\frac{1}{P_{AF}}v^F_{a_1}\frac{1}{P_{AF}}v^F_{a_2}...\frac{1}{P_{AF}}v^F_{a_N}\delta^F_{c})\bigg), \quad (5.22)\end{aligned}$$

where δ^F_{rc} and δ^F_c stand for

$$\delta^F_{rc} = \frac{i\epsilon_{rpq}}{2b}\frac{1}{P_{AF}}\sigma_p x^F_q\frac{1}{1+a^2F^F}\hat{\Gamma}^L\sigma_c\ ,\ \delta^F_c = \frac{a^2}{4b^3}\frac{1}{P_{AF}}F^F\frac{1}{1+a^2F^F}\hat{\Gamma}^L\sigma_c. \quad (5.23)$$

Before we take the continuum limit of these expressions we remark that the components of the two vectors $\vec{\delta}_2 x^F$ and $\vec{\delta}_3 x^F$ are matrices in Mat_{2l+1} which are covariant under gauge transformations. Furthermore because of the presence of the traces $Tr_l(n^F_{a_N}...n^F_{a_1}n^F_r)$ and $Tr_l(n^F_{a_N}...n^F_{a_1})$, the components $\delta_2 x^F_c$ and $\delta_3 x^F_c$ are also invariant under the extra symmetry $n^F_a{\longrightarrow}w^F_a = W^+ n^F_a W$ where W is an arbitrary unitary transformation given by $W = e^{i\alpha^F}$, $\alpha^F{\in}Mat_{2l+1}$. These transformations W are different from gauge transformations U introduced in section 3 . For example one can see from their explicit action

$$n^F_a{\longrightarrow}w^F_a = W^+ n^F_a W = n^F_a + \frac{1}{\sqrt{l(l+1)}}W^+[L_a, W], \quad (5.24)$$

that the fuzzy coordinates w_a^F reduce to the same continuum coordinates n_a in the limit, in other words this extra symmetry is not available in the continuum . Using this symmetry one can therefore rewrite the vectors $\delta_2 x_c^F$ and $\delta_3 x_c^F$ in the equivalent form

$$\delta_2 x_c^F = \sum_{N=1}\bigg(\frac{1}{2l+1}Tr_l(w_{a_N}^F...w_{a_1}^F w_r^F)\bigg)\bigg(tr_2(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F...\frac{1}{P_{AF}}v_{a_N}^F\delta_{rc}^F)\bigg) \quad (5.25)$$

and

$$\delta_3 x_c^F = \sum_{N=0}\bigg(\frac{1}{2l+1}Tr_l(w_{a_N}^F...w_{a_1}^F)\bigg)\bigg(tr_2(\frac{1}{P_{AF}}v_{a_1}^F\frac{1}{P_{AF}}v_{a_2}^F...\frac{1}{P_{AF}}v_{a_N}^F\delta_{c}^F)\bigg). \quad (5.26)$$

We can now immediately write down the continuum limit of the above expressions . We already know that in this limit $a\longrightarrow 0$, $b\longrightarrow 1$, $\frac{1}{2l+1}Tr_l\longrightarrow\int_{\mathbf{S}^2}$, $n_a^F\longrightarrow n_a$, $x_a^F\longrightarrow n_a$, $F_{ab}^F\longrightarrow F_{ab}$, $\lambda^F\longrightarrow\lambda$, $\alpha^F\longrightarrow\alpha$ and $\hat{\Gamma}^L\longrightarrow\gamma$. From equation (5.10) we also see that in this limit $P_{AF}\longrightarrow 1$ and $v_a^F\longrightarrow n_a$, whereas from equation (5.24) we see that $w_a^F\longrightarrow n_a$. For example the continuum limit of (5.21) is given by

$$\delta_2\mathcal{A}_\theta = -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})\delta_2 x_c(\vec{n})$$
$$\delta_2 x_c(\vec{n}) = \frac{i}{2}\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}(\vec{n}\times\vec{n}^{'})_p tr_2\bigg(\sigma_c\sum_{N=1}(\vec{n}.\vec{n}^{'})^N\sigma_p\gamma(\vec{n})\bigg). \quad (5.27)$$

It is easily seen that all the expected continuum divergence arises from the series $\sum_{N=1}(\vec{n}.\vec{n}^{'})^N$ when $\vec{n}=\vec{n}^{'}$, i.e from contact terms . Thus we need first to separate contact terms as follows

$$\delta_2 x_c(\vec{n}) = \frac{i}{2}\bigg[(\vec{n}\times\vec{n}^{'})_p tr_2\bigg(\sigma_c\sum_{N=1}(\vec{n}.\vec{n}^{'})^N\sigma_p\gamma(\vec{n})\bigg)\bigg]_{\vec{n}^{'}=\vec{n}} + \delta_2 x_c(\vec{n})|_{nct} \quad (5.28)$$

where

$$\delta_2 x_c(\vec{n})|_{nct} = \frac{i}{2}\int_{\vec{n}^{'}\neq\vec{n}}\frac{d\Omega^{'}}{4\pi}(\vec{n}\times\vec{n}^{'})_p tr_2\bigg(\sigma_c\sum_{N=1}(\vec{n}.\vec{n}^{'})^N\sigma_p\gamma(\vec{n})\bigg). \quad (5.29)$$

It is clear that in $\delta_2 x_c^F(\vec{n})|_{nct}$ we have $|\vec{n}.\vec{n}^{'}|=|cos\theta|<1$ where θ is the angle between $\vec{n}$ and $\vec{n}^{'}$ and thus the series $\sum_{N=1}(\vec{n}.\vec{n}^{'})^N$ is now convergent . We can therefore simply substitute the result

$$\sum_{N=1}(\vec{n}.\vec{n}^{'})^N = \frac{\vec{n}.\vec{n}^{'}}{1-\vec{n}.\vec{n}^{'}},$$

and thus obtain

$$\delta_2 x_c(\vec{n})|_{nct} = \int_{\vec{n}^{'}\neq\vec{n}}\frac{d\Omega^{'}}{4\pi}\bigg(n_c(\vec{n}.\vec{n}^{'})-n_c^{'}\bigg)\frac{\vec{n}.\vec{n}^{'}}{1-\vec{n}.\vec{n}^{'}}$$
$$= (n_c n_d-\delta_{cd})n_e\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}\frac{n_d^{'}n_e^{'}}{1+\epsilon-\vec{n}.\vec{n}^{'}},$$

where in the second line we have replaced $\int_{\vec{n}'\neq\vec{n}}\frac{d\Omega^{'}}{4\pi}$ by the intgeral over the full sphere $\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}$ with the prescription $\epsilon>0$ so that the contribution of the terms $\vec{n}^{'}=\vec{n}$ is kept equal to zero .

Remark that $n_c\delta_2x_c(\vec{n})|_{nct}=0$ and hence the contribution of the above non-contact terms is identically zero since in the continuum we also have $n_aA_a=0$, $\epsilon_{abc}F_{ab}=2in_c\partial_b(A_b)$ and thus

$$\begin{aligned}\delta_2\mathcal{A}_\theta|_{nct} &= -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})\delta_2x_c(\vec{n})|_{nct}\\ &= 4\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda\partial_b(A_b)n_c\delta_2x_c|_{nct}\\ &\equiv 0.\end{aligned}\tag{5.30}$$

For contact terms the series $\sum_{N=1}(\vec{n}.\vec{n}^{'})^N$ is obviously divergent and thus the first term of equation (5.28) is not well defined . Thus it is natural to go back to the finite expression (5.25) and think of it now as a gauge-invariant regularization of contact terms with a cut-off l provided here by the fuzzy sphere . Since we are interested in the continuum theory the cut-off l is effectively large and thus one may retain only the first few corrections to the continuum theory . As it turns out the contribution of contact terms is regularized completely and in a gauge-invariant manner if we only keep $O(\frac{1}{l})$ corrections. To see how all this works explicitly , we use the star product on $\mathbf{S}^2_F$ and replace $\delta_2x_c^F$ by its image $<\delta_2x_c^F>(\vec{n})$ via the coherent states , namely

$$\begin{aligned}<\delta_2x_c^F>(\vec{n}) =& \sum_{N=1}\bigg(\int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}<w^F_{a_N}>*...*<w^F_{a_1}>*<w^F_r>(\vec{n}^{'})\bigg)\\ &\times\bigg(tr_2<\frac{1}{P_{AF}}>*<v^F_{a_1}>*<\frac{1}{P_{AF}}>*<v^F_{a_2}>*...*<\frac{1}{P_{AF}}>*<v^F_{a_N}>*<\delta^F_{rc}>(\vec{n})\bigg).\end{aligned}$$

Since non-contact terms are regular in the limit one can still treat them in the same way as before and hence their contribution is zero . This means that $<\delta_2x_c^F>(\vec{n})$ is dominated by contact terms , namely

$$\begin{aligned}<\delta_2x_c^F>(\vec{n}) =& \sum_{N=1}\bigg(<w^F_{a_N}>*...*<w^F_{a_1}>*<w^F_r>(\vec{n})\bigg)\\ &\times\bigg(tr_2<\frac{1}{P_{AF}}>*<v^F_{a_1}>*<\frac{1}{P_{AF}}>*<v^F_{a_2}>*...*<\frac{1}{P_{AF}}>*<v^F_{a_N}>*<\delta^F_{rc}>(\vec{n})\bigg).\end{aligned}$$

From equation (2.19) we have $<n_a^F>=n_a-\frac{1}{2l}n_a+O(\frac{1}{l^2})$ and thus we can compute to this order in $\frac{1}{l}$ the following quantities

$$\begin{aligned}&<\hat{\Gamma}^L>=\gamma+\frac{1}{l}(-\frac{1}{2}\gamma+\sigma A+\frac{1}{2})+O(\frac{1}{l^2})\ ,\ \ <\frac{1}{P_{AF}}>=1-\frac{1}{2l}\gamma+O(\frac{1}{l^2})\\ &<v_a^F>=n_a+\frac{1}{l}(-\frac{1}{2}n_a+A_a+\frac{\sigma_a}{2})+O(\frac{1}{l^2})\ ,\ \ <w_a^F>=n_a+\frac{1}{l}(-\frac{1}{2}n_a+i\mathcal{L}_a(\alpha))+O(\frac{1}{l^2})\\ &<\delta^F_{rc}>=\frac{i}{2}\epsilon_{rpq}\bigg(n_q\sigma_p\gamma\sigma_c+\frac{1}{l}\sigma_p(-n_q\gamma+n_q\sigma A+\frac{1}{2}n_q+\gamma A_q)\sigma_c-\frac{1}{2l}n_q\gamma\sigma_p\gamma\sigma_c\bigg)+O(\frac{1}{l^2}).\end{aligned}$$

Also by using the star product on $\mathbf{S}^2_F$ we compute

$$<\frac{1}{P_{AF}}>*<v_a^F>=<\frac{1}{P_{AF}}><v_a^F>+O(\frac{1}{l^2})=n_a+\frac{1}{l}\bigg(-\frac{1}{2}n_a+A_a+\frac{\sigma_a}{2}-\frac{1}{2}\gamma n_a\bigg)+O(\frac{1}{l^2}).$$

Remark for example the continuum limit $\delta^F_{rc}\longrightarrow\delta_{rc}=\frac{i}{2}\epsilon_{rpq}\sigma_p n_q\gamma\sigma_c$ when $l\longrightarrow\infty$ as well as the limit $<\frac{1}{P_{AF}}>*<v_a^F>\longrightarrow n_a$ when $l\longrightarrow\infty$. Finally and by using once again the star product we compute the formulae

$$\begin{aligned}<w^F_{a_N}>*...*<w^F_{a_1}>*<w^F_r>(\vec{n})&=<w^F_{a_N}>...<w^F_{a_1}><w^F_r>\\&+\frac{1}{2l}K_{pq}n_r\sum_{i=2}^{N}\sum_{j<i}n_{a_N}...\partial_p n_{a_i}...\partial_q n_{a_j}...n_{a_1}\\&+\frac{1}{2l}K_{pr}\partial_p(n_{a_N}...n_{a_1})+O(\frac{1}{l^2}),\end{aligned}$$

and

$$\begin{aligned}<\frac{1}{P_{AF}}>*<v^F_{a_1}>*...*<\frac{1}{P_{AF}}>*<v^F_{a_N}>*<\delta^F_{rc}>&=<\frac{1}{P_{AF}}><v^F_{a_1}>...<\frac{1}{P_{AF}}><v^F_{a_N}><\delta^F_{rc}>\\&+\frac{1}{2l}\delta_{rc}K_{pq}\sum_{i=1}^{N-1}\sum_{j>i}n_{a_1}...\partial_p n_{a_i}...\partial_q n_{a_j}...n_{a_N}\\&+\frac{1}{2l}K_{pq}\partial_p(n_{a_1}...n_{a_N})\partial_q\delta_{rc}+O(\frac{1}{l^2}).\end{aligned}$$

Putting all these results together we obtain the contribution of contact terms in the form

$$\begin{aligned}<\delta_2 x^F_c>(\vec{n})&=tr_2\bigg(\sum_{N=1}\bigg(<\frac{1}{P_{AF}}><v^F_a><w^F_a>\bigg)^N<\delta^F_{rc}><w^F_r>\bigg)\\&=(l+1)\bigg(tr_2<\delta^F_{rc}>\bigg)<w^F_r>,\end{aligned}$$

where we have also used the result

$$\sum_{N=1}\bigg(<\frac{1}{P_{AF}}><v^F_a><w^F_a>\bigg)^N=\sum_{N=1}(1-\frac{1}{l})^N=l+1.$$

In other words the sum over N is linearly divergent with l and hence the remaining term $(tr_2<\delta^F_{rc}>)<w^F_r>$ must approach in the continuum limit 0 at least as $\frac{1}{l}$ in order to reproduce a finite expression. A final calculation shows that this is indeed the case where from the equation

$$\begin{aligned}tr_2<\delta^F_{rc}>&=\frac{i}{2}\epsilon_{rpq}n_q tr_2\sigma_p\gamma\sigma_c+\frac{i\epsilon_{rpq}}{2l}tr_2\sigma_c\sigma_p(-n_q\gamma+n_q\sigma A+\frac{1}{2}n_q+\gamma A_q)-\frac{i\epsilon_{rpq}}{4l}n_q tr_2\gamma\sigma_p\gamma\sigma_c\\&=(n_r n_c-\delta_{rc})+\frac{i\epsilon_{rcq}n_q}{l}-\frac{1}{l}[n_r n_c-\delta_{rc}-n_c A_r-n_r A_c],\end{aligned}$$

we deduce that $(tr_2<\delta^F_{rc}>)<w^F_r>=\frac{1}{l}[A_c-i\mathcal{L}_c(\alpha)]$. The contribution of contact terms is therefore finite equal to

$$<\delta_2 x^F_c>(\vec{n})=A_c-i\mathcal{L}_c(\alpha), \tag{5.31}$$

and as a consequence

$$\delta_2\mathcal{A}_\theta = -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})A_c(\vec{n}). \tag{5.32}$$

The dependence on the arbitrary function α drops because of the identity $\epsilon_{abc}F_{ab}\mathcal{L}_c(\alpha)=0$. This is equivalent to the fact that the term $-i\mathcal{L}_c(\alpha)$ in (5.31) can always be canceled by a continuum gauge transformation $A_a^{'}=A_a-i\mathcal{L}_a(\Omega)$, i.e the fuzzy symmetries (5.24) are not available as expected in the continuum setting since they can be eliminated by gauge symmetries . (5.32) is exactly the topological Chern-Simons action in two dimensions and thus it vanishes identically as one might easily check. Its emergence from contact terms of the propagator $\frac{1}{\mathcal{D}_{aA}}$ is intimately related to the fact that the top modes of the Dirac operator $\mathcal{D}_{Aa}$ as opposed to those of the exact Diarc operator $\mathcal{D}_{AF}$ are very large .

Similarly to above the last correction to the anomaly given by equation (5.22) has the limit

$$\begin{aligned}\delta_3\mathcal{A}_\theta &= -2i\epsilon_{abc}\int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\lambda(\vec{n})F_{ab}(\vec{n})\delta_3 x_c(\vec{n})\\ \delta_3 x_c(\vec{n}) &= \int_{\mathbf{S}^2}\frac{d\Omega^{'}}{4\pi}tr_2\bigg(\sum_{N=0}(\vec{n}.\vec{n}^{'})^N\delta_c(\vec{n})\bigg).\end{aligned} \tag{5.33}$$

The vector $\vec{\delta}(\vec{n})$ in above is in fact identically zero as one might easily check . Again for non-contact terms the series $\sum_{N=0}(\vec{n}.\vec{n}^{'})^N$ converges and thus the corresponding contribution clearly vanishes . For contact terms we have to regularize as before but since now the vector $<\vec{\delta}^F>(\vec{n})$ vanishes as $\frac{1}{l^2}$ and not as $\frac{1}{l}$ we immediately conclude that the contribution of contact terms is zero in this case and thus we obatin the result

$$\delta_3\mathcal{A}_\theta \equiv 0. \tag{5.34}$$

6. Conclusion

We showed that the local axial anomaly in $2-$dimensions emerges naturally if one postulates an underlying noncommutative fuzzy structure of spacetime . As we have discussed in this article this result consists in three main parts :

1) We showed that the Dirac-Ginsparg-Wilson relation on $\mathbf{S}^2_F$ contains already at the classical level the anomaly in the form of an edge effect which under quantization becomes precisely the "fuzzy" $U(1)_A$ axial anomaly on the fuzzy sphere.

2) We derived a novel gauge-covariant expansion of the quark propagator in the form $\frac{1}{\mathcal{D}_{AF}}=\frac{a\hat{\Gamma}^L}{2}+\frac{1}{\mathcal{D}_{Aa}}$ where $a=\frac{2}{2l+1}$ is the lattice spacing on $\mathbf{S}^2_F$, $\hat{\Gamma}^L$ is the covariant noncommutative chirality and $\mathcal{D}_{Aa}$ is an effective Dirac operator which has essentially the same IR spectrum as $\mathcal{D}_{AF}$ but differes from it on the UV modes since the eigenvalues of $\mathcal{D}_{Aa}$ on the top modes are very large compared to those of $\mathcal{D}_{AF}$. Most remarkably however is the fact that both operators share the same continuum limit and thus the above covariant expansion is not available in the continuum theory .

3) The first bit in this expansion $\frac{a\hat{\Gamma}^L}{2}$ although it vanishes as it stands in the continuum limit , its contribution to the anomaly is exactly the canonical theta term. The contribution

of the propagator $\frac{1}{\mathcal{D}_{Aa}}$ is on the other hand equal to the toplogical Chern-Simons action which in two dimensions is identically zero . In particular we have explicilty shown that beside the cut-off l provided by the star product of the fuzzy sphere itself there is no need to any extra regulator even while approaching the limit .

Finally we have to note that a complete extension of the above results to the case of monopoles is more or less straightforward after we identify correctly the corresponding bundle structure on $\mathbf{S}^2_F$. As it turns out this is indeed possible and thus the extension can be made without much difficulty . The relevant detail will be however reported elsewhere. The computation of the effective action of the Schwinger model on the fuzzy two-sphere, which will allow us on the other hand to probe the solvability of the model , will also be reported elsewhere .

Acknowledgments

I would like to thank A.P.Balachandran , G.Immirzi , Denjoe O'Connor and P.Presnajder for their critical comments while the work was in progress. This work was supported in part by the DOE under contract number DE-FG02-85ER40231.

7. Appendix

1 On continuum $\mathbf{S}^2$ exact chiral invariance of the free classical action is expressed by the anticommutation relation $\{\gamma,\mathcal{D}\}=0$ which is in fact the limit of the Ginsparg-Wilson relation (3.10). In the presence of gauge fields , the Ginsparg-Wilson relation (3.10) becomes (4.11) and the edge effect seen there is exactly the source of the anomaly . We will now show that if we try instead to formulate chiral invariance on the fuzzy sphere without edge effect then the action becomes complex and not gauge invariant and thus not quantizable in any canonical fashion .

As we have already said the continuum limit of the classical actions (3.5) and (3.12) are given respectively by (3.13) and (3.14) . (3.14) was already shown to be gauge invariant, but under the canonical continuum chiral transformations $\psi\longrightarrow\psi^{'}=\psi+\lambda\gamma\psi$, $\bar{\psi}\longrightarrow\bar{\psi}^{'}=\bar{\psi}+\lambda\bar{\psi}\gamma$, one can show that it is invariant only if the gauge field is constrained to satisfy $n_aA_a=0$ because of the identity $\mathcal{D}_A\gamma+\gamma\mathcal{D}_A=2\vec{A}.\vec{n}$. From the continuum limit of (3.4) it is obvious that this constraint is satisfied and hence chiral symmetry is maintained. However one wants also to formulate chiral symmetry without the need to use any constraint on the gauge field , indeed the action

$$\int\frac{d\Omega}{4\pi}\bigg[\bar{\psi}\mathcal{D}\psi+\bar{\psi}\hat{\sigma}_aA_a\psi\bigg]. \tag{7.1}$$

is strictly chiral invariant for arbitrary gauge configurations . $\hat{\sigma}_a$ is the Clifford algebra projected onto the sphere , i.e $\hat{\sigma}_a=\mathcal{P}_{ab}\sigma_b$, $\mathcal{P}_{ab}=\delta_{ab}-n_an_b$. The action (7.1) is also still gauge invariant because of the identity $n_a\mathcal{L}_a=0$. Action (7.1) can be rewritten as follows

$$S_C=\int\frac{d\Omega}{4\pi}\bigg[\bar{\psi}\mathcal{D}\psi+\epsilon_{abc}\bar{\psi}Z_bn_cA_a\psi\bigg]\ ,\ Z_a=\frac{i}{2}[\gamma,\sigma_a]. \tag{7.2}$$

From this form of the action one can immediately define the following chiral-covariant Dirac operator $\mathcal{D}_C = \mathcal{D} + \epsilon_{abc}Z_b n_c A_a$. The difference between the continuum gauge-covariant Dirac operator $\mathcal{D}_A$ and the continuum chiral-covariant Dirac operator $\mathcal{D}_C$ is proportional to the normal component $\phi = n_a A_a$ of the gauge field , i.e $\mathcal{D}_C = \mathcal{D}_A - \gamma\phi$, and hence they are essentially identical (by virtue of the constraint $n_a A_a = 0$) , i.e (3.14) and (7.2) are equivalent actions . The Fuzzy analogue of (3.14) is (3.12) whereas the fuzzy analogue of (7.2) is the action

$$S_{CF} = \frac{1}{2l+1}Tr_l\Big[\bar{\psi_F}\mathcal{D}_F\psi_F + \epsilon_{abc}\bar{\psi_F}Z_b^F n_c^F A_a^F\psi_F\Big] \ , \ Z_a^F = \frac{i}{2}[\Gamma^L\sigma_a + \sigma_a\Gamma^R]. \quad (7.3)$$

The underlying fuzzy Dirac operator is obviously given by $\mathcal{D}_{CF} = \mathcal{D}_F + \epsilon_{abc}Z_b^F n_c^F A_a^F$ which tends in the large l limit to $\mathcal{D}_C$. This means in particular that the two Dirac operators $\mathcal{D}_{CF}$ and $\mathcal{D}_{AF}$ have the same continuum limit. As a consequence the corresponding fuzzy classical actions (7.3) and (3.12) approach the same continuum action (3.14) in the limit. Remark also that the Dirac operator $\mathcal{D}_{CF}$ satisfies the Ginsparg-Wilson relation $\mathcal{D}_{CF}\Gamma^R - \Gamma^L\mathcal{D}_{CF} = 0$ and hence the action (7.3) is invariant under free fuzzy chiral transformations

$$\begin{aligned} &\psi_F \longrightarrow \psi_F^{'} = \psi_F + \Gamma^R\psi_F\lambda^F + O((\lambda^F)^2) \\ &\bar{\psi}_F \longrightarrow \bar{\psi}_F^{'} = \bar{\psi}_F - \lambda^F\bar{\psi}_F\Gamma^L + O((\lambda^F)^2). \end{aligned} \quad (7.4)$$

In here the chiral parameter λ^F which is still a general $(2l+1)\times(2l+1)$ matrix does not transform under gauge symmetries in contrast with the case considered in section 4. Also these fuzzy chiral transformations (7.4) as opposed to the fuzzy chiral transformations (4.14) do not depend on the gauge field , yet both (7.4) and (4.14) reduce in the limit to the same continuum chiral transformations .

Remark however that the Dirac operator $\mathcal{D}_{CF}$ is not self-adjoint as well as not gauge-covariant. In other words the action (7.3) is complex and not gauge invariant and thus it is not suited for any quantization procedure.

2 **Measure-Transforming Chiral Symmetries** In the remainder of this appendix we introduce measure-changing chiral transformations as opposed to the chiral transformations (4.14) which leave the measure invariant. These new transformations as it turns out yield the same anomaly (4.21) and thus they are equivalent to the transformations (4.14) . The only difference is the fact that this anomaly (4.21) arises now from the measure instead. We will also define gauge-invariant axial current as opposed to the gauge-covriant axial current defined in section 4.3.

Remark that since $\Gamma^R - \hat{\Gamma}^L \longrightarrow -2\gamma$ when $l \longrightarrow \infty$, the continuum limit of (4.18) is the usual formal answer, i.e

$$S_\theta = -\int\frac{d\Omega}{4\pi}\lambda(x)\sum_\mu \phi_\mu^+(x)(-2\gamma)\phi_\mu(x). \quad (7.5)$$

In the continuum the sum $\sum_\mu$ is not cutoff since all orbital angular momenta are allowed and obviously $\phi_\mu(x)$ stands now for the eigenfunctions of the continuum gauged Dirac operator $\mathcal{D}_A = \mathcal{D} + \sigma_a A_a$. These states satisfy the completeness relation

$$\sum_{\mu} \phi_{\alpha}(x)\phi^{+}_{\beta}(y) = \delta^2(x-y)\delta_{\alpha\beta}. \tag{7.6}$$

If we try now to use this completeness relation in (7.5) we will instead get the ill-defined expression $S_{\theta} = 2\int \frac{d\Omega}{4\pi}\lambda(x)\delta^2(0)tr(\gamma)$ and thus a regularization is needed . For example Fujikawa regularization of (7.5) was performed in [16] and was shown there to reproduce the correct anomaly . Here however we know from (4.19) that the action (4.18) on this finite matrix model is identically zero and this can not be made into anything else which seems in contradiction with Fujikawa regularization of (7.5) .

It is therefore natural to modify appropriately chiral transformations in such a way as to shift the anomaly (4.21) from the action to the measure . The required deformation is not difficult to find and one obtains

$$\begin{aligned} \psi_F &\longrightarrow \psi_F^{'} = \psi_F + (\Gamma^R\lambda\psi_F) + O(\lambda^2) \\ \bar{\psi}_F &\longrightarrow \bar{\psi}_F^{'} = \bar{\psi}_F - (\bar{\psi}_F\lambda\hat{\Gamma}^L) - (\bar{\psi}_F\lambda\delta\Gamma) + O(\lambda^2), \end{aligned} \tag{7.7}$$

where the deformation $\delta\Gamma$ is given by

$$\delta\Gamma = -\frac{i}{2l+1}\epsilon_{abc}\sigma_c F^F_{ab}\frac{1}{\mathcal{D}_{AF}}. \tag{7.8}$$

Remark that (7.7) is also dictated by (4.11) which can be put in the equivalent form $\mathcal{D}_{GF}\Gamma^R - (\hat{\Gamma}^L + \delta\Gamma)\mathcal{D}_{GF} = 0$. Remark also that all needed requirement are satisfied by this deformation : $\delta\Gamma$ is gauge covariant and drops in the limit and therefore the transformations (7.7) are consistent with gauge symmetry and reduce in the limit to ordinary chiral transformations . Remark also that we can invert the gauged Dirac operator since we are considering only trivial $U(1)-$bundles over $\mathbf{S}^2$, i.e there is no monopole. Under these modified chiral transformations the change in the action is only a total covariant divergence , namely

$$\Delta S_{AF} = -\frac{1}{2l+1}Tr_l\lambda[D^F_a, \mathcal{J}^5_a], \tag{7.9}$$

while the measure becomes non-symmetric , in other words

$$\begin{aligned} S_{\theta F} &= -\frac{1}{2l+1}Tr_l\sum_{\mu}\phi^{+}(\mu,A)\lambda(\Gamma^R - \hat{\Gamma}^L - \delta\Gamma)\phi(\mu,A) \\ &= -\frac{i}{2l+1}\epsilon_{abc}(\lambda)^{BC}(F^F_{ab})^{CD}tr_2\sigma_c(\mathcal{D}^{-1}_{AF})^{DA,BA}. \end{aligned} \tag{7.10}$$

This anomaly is of course exactly identical to the result (4.22) since what we have just done is to shift the anomaly from the action to the measure . Remark also that since $\Gamma^R - \hat{\Gamma}^L - \delta\Gamma \longrightarrow -2\gamma$ when $l\longrightarrow\infty$ the continuum limit of the first line of (7.10) is still given by (7.5) which as we said needs a proper regularization, whereas the second line of (7.10) can now be thought of as a regularization of (7.5) which is provided in this noncommutative context by the fuzzy sphere .

3 **Gauge-Invariant Axial Current** Similarly to the gauge-covariant axial current defined in section 4.3 we define now gauge-invariant axial current . First we introduce the following chiral transformations

$$\begin{aligned}
&\psi_F \longrightarrow \psi_F^{'} = \psi_F + (\Gamma^R\psi_F)\lambda + O(\lambda^2) \\
&\bar{\psi}_F \longrightarrow \bar{\psi}_F^{'} = \bar{\psi}_F - \lambda(\bar{\psi}_F\hat{\Gamma}^L) - \lambda(\bar{\psi}_F\delta\Gamma) + O(\lambda^2),
\end{aligned} \tag{7.11}$$

where now the chiral parameter λ is a matrix in Mat_{2l+1} which does not transform under gauge transformations. Under these gauge-invariant chiral transformations the change in the action is only a total divergence , namely

$$\begin{aligned}
\Delta S_{GF} &= -\frac{1}{2l+1}Tr_l\lambda[L_a, J_a^5], \\
J_a^5 &= \bar{\psi}_F\sigma_a\Gamma^R\psi_F.
\end{aligned} \tag{7.12}$$

The measure is still non-symmetric but its change is now of the form

$$\begin{aligned}
S_{\theta F} &= -\frac{1}{2l+1}Tr_l\lambda\sum_\mu \phi^+(\mu, A)(\Gamma^R - \hat{\Gamma}^L - \delta\Gamma)\phi(\mu, A) \\
&= -\frac{i}{2l+1}\epsilon_{abc}(\lambda)^{BA}(F_{ab}^F)^{CD}tr_2\sigma_c(\mathcal{D}_{AF}^{-1})^{DB,CA}.
\end{aligned} \tag{7.13}$$

The contribution of $\Gamma^R - \hat{\Gamma}^L$ vanished by a similar argument to that which led to (4.19) . The WT identity $\Delta S_{GF} = S_{\theta F}$ will now simply look like

$$[L_a, J_a^5]^{AB} = i\epsilon_{abc}(F_{ab}^F)^{CD}tr_2\sigma_c(\mathcal{D}_{AF}^{-1})^{DB,CA}. \tag{7.14}$$

In this case since $[L_a, J_a^5]^{AB}$ does not transform under gauge transformations , one can immediately conclude that the left hand side must also not transform under gauge transformations and therefore it can only be proportional to the identity , i.e $[L_a, J_a^5]^{AB}\propto\delta^{AB}$.

References

[1] A.Connes , "Noncommutative Geometry" , Academic Press, London , 1994 .

[2] J. Madore. *An Introduction to Noncommutative Differential Geometry and its Applications.* Cambridge University Press, Cambridge, 1995; gr-qc/9906059.

[3] H. Grosse, C. Klimcik, P. Presnajder. hep-th/9602115; *Commun.Math.Phys.* **180**(1996)429-438. H. Grosse, C. Klimcik, P. Presnajder. hep-th/9505175; *Int.J.Theor.Phys.* **35**(1996)231-244.

[4] Badis Ydri , *Fuzzy Physics* , hep-th/0110006 , Ph.D Thesis .

[5] D.A.Varshalovich , A.N.Moskalev , V.K.Khersonky , "Quantum Theory of Angular Momentum : Irreducible Tensors , Spherical Harmonics , vector Coupling Coefficients, 3nj Symbols " , Singapore , Singapore : World Scientific (1998) .

[6] P.Prešnajder , hep-th/9912050 and *J.Math.Phys.* **41**(2000)2789-2804.

[7] A.P.Balachandran , T.R.Govindarajan , B.Ydri , hep-th/9911087; A.P.Balachandran , T.R.Govindarajan , B.Ydri , hep-th/0006216 and *Mod.Phys.Lett.***A15** , 1279 (2000).

[8] S.Baez , A.P. Balachandran , S.Vaidya and B.Ydri , hep-th/9811169 and Comm.Math.Phys.**208**,787(2000).

[9] H. Grosse and P. Prešnajder , *Lett.Math.Phys.***33**, 171-182(1995). H. Grosse, C. Klimčík and P. Prešnajder ,*Commun.Math.Phys.***178**, 507-526 (1996). H. Grosse, C. Klimčík and P. Prešnajder in *Les Houches Summer School on Theoretical Physics*, 1995, hep-th/9603071 .

[10] U.Carow-Watamura and S.Watamura ,hep-th/9605003,*Comm.Math.Phys.***183**,365 (1997). U.Carow-Watamura and S.Watamura,hep-th/9801195,*Comm.Math.Phys.* **212**,395(2000).

[11] D.Karabali , V.P.Nair and A.P.Polychronakos , hep-th/0111249 .

[12] A.P.Balachandran , B.P.Dolan , J.Lee , X.Martin and D.O'Connor , hep-th/0107099 , J.Geom.Phys. 43(2002)184-204.

[13] K.Fujikawa ,*Phys.Rev.Lett.***42**,1195(1979) ; *Phys.Rev.***D21**,2848(1980).

[14] N.B.Nielsen , M.Ninomiya ,*Phys.Lett.***B105**,219(1981);*Nucl.Phys.***B185**,20(1981).

[15] A.P.Balachandran and S.Vaidya , Int.J.Mod.Phys.**A16**,17(2001) .

[16] H. Aoki, S. Iso and K. Nagao, "Ginsparg-Wilson relation, topological invariants and finite noncommutative geometry" , hep-th/0209223, H. Aoki, S. Iso and K. Nagao, "Chiral anomaly on fuzzy 2-sphere", hep-th/0209137.

[17] Giorgio Immirzi , Badis Ydri ; " Chiral Symmetry on $\mathbf{S}^2_F$" , hep-th/0203121. Proceeding of The Theoretical High-Energy Physics Conference At Utica, New York , 12 Oct 2001 , edited by A.H.Fariborz and E.Rusjan.

[18] J . Nishimura , M .A .Vazquez-Mozo , hep-th/0107110(JHEP 0108 (2001) 033) and hep-lat/0210017 .

[19] A.P.Balachandran , G.Immirzi , "The Fuzzy Ginsparg-Wilson Algebra : A Solution of the Fermion Doubling Problem" , hep-th/0301242 .